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Black Hole Spectrum, Horison Quantization and All That: $(2+1)$-Dimensional Example .

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Abstract

We discuss the possibility that quantum black holes have discrete mass spectrum. Different arguments leading to this conclusion are considered, particularly the decoupling between left and right sectors in string theory - the so-called heterotic principle. The case of a $2+1$ dimesnional black holes is considered as an argument in favour of this argument. The possible connection between membrane model of the black hole horison and topological membrane is briefly discussed.

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1. Introduction.

Quantum black holes play the same role now as black body radiation did a century ago, creating problems and paradoxes which the future theory (presumably of the twenty-first century) will resolve. The fact that the most intriguing process occuring with quantum black holes, Hawking quantum evaporation [1], describes the outcoming radiation as having (almost) the black body spectrum, supports in an amusing way a continuity of tradition in theoretical physics. Using this as a guideline one can ask the question whether the quantum black hole is an analog of another cornerstone of twentieth-century quantum physics - the hydrogen atom. Do quantum black holes have a discrete spectrum and if yes - may it be that the Hawking evaporation is the transition process between the discrete states ?

The idea that black holes may have a discrete spectrum was first proposed by Bekenstein in 1974 [2] who used an analogy between a horison area A for Kerr black hole, proportional to the squared irreducible mass $M_{ir}^2 = A/16\pi$, and an action integral $\oint pdq$ of a periodic mechanical system. This analogy was based on a fact that the irreducible mass behaves as an adiabatic invariant, i.e. remains unchanged in reversible processes (existing only for Kerr black hole with non-zero angular momentum, see [2] for details). Using this analogy and Bohr-Sommerfeld quantization condition Bekenstein obtained the discrete spectrum

$$M_{ir}^2 \sim M_p^2 n \tag{1.1}$$

where M_p is the Planck mass and n is an integer. Later, in 1986 V.Mukhanov [3] and author[4] * independently revived this idea using completely different arguments, which

*Unfortunately at that time I was unaware about Bekenstein paper [2] as well as now have no information about any paper discussing this subject between 1974 and 1986

will be briefly discussed later. The last year this problem attracted more attention and has been discussed in several interesting papers [5] - [9], where the discrete spectrum was derived using new ideas. Let us also note that quantization of the area operator in quantum gravity has been obtained in [10], [11] using the loop representation.

Recently the new class of black holes in three-dimensional spacetime with a negative cosmological constant was considered by Banados, Teitelboim and Zanelli [12] (see also detailed paper [13]). The particular case of charged black holes in the limit of zero cosmological constant as well as black hole solutions in $U(1)$ topologically massive gauge theory (i.e. $2+1$ gauge theory with a Chern-Simons term) has been considered at the same time in [14].

In two recent papers Maggiore [15] and Carlip [16] studied the entropy of neutral $2+1$ black holes [12], [13], [17]. In [15] the membrane approach, which has been previously applied to $3+1$ black holes [7], [9], reproduced the correct value of entropy. The Schroedinger equation for the horizon wave function has been written, which leads to the discrete spectrum (however the explicit form of the spectrum has not been discussed there). In [16] the similar picture was obtained, where the "membrane" degrees of freedom at the horison were nothing but the gauge degrees of freedom arising due to the breaking of gauge invariance of $2+1$-dimensional gravity due to the presence of horizon. The entropy was obtained as the logarithm of the number of these states. But in obtaining this the relation between the horizon radius r_+ and a number operator N with integer spectrum has been obtained

$$N = \left(\frac{r_+}{4G}\right)^2 \tag{1.2}$$

which means the quantum spectrum for mass $M = r_+^2/l^2$, where l is an inverse cosmolog-

ical constant, takes the following form

$$M = \left(\frac{4G}{l}\right)^2 N \tag{1.3}$$

Let us note that it is proportional to N and not to $\sqrt{N}$ as in $3+1$ case.

The aim of this paper is to discuss this spectrum as well as spectrum arising for charged black holes and find what approach used previously lead to this spectrum. In the next section we shall discuss arguments used in [3] and [4]. Than we shall remind some basic facts about 2+1 black holes and briefly discuss the Carlip approach [16] leading, as we shall demonstrate, to the spectrum (1.2). Generalization of this quantization condition in case of black holes discussed in [14] will be given. Because entropy and action both proportional to $r_+ \sim \sqrt{N}$ [12] it is not so easy to get this spectrum from Bohr-Sommerfeld quantization. However this spectrum is in agreement with string-inspired arguments presented in [4]. In conclusion we shall discuss possible connection between these arguments and Maggiore membrane approach [7],[9], [15].

2. Heuristic arguments for black hole quantization.

Now we shall consider arguments presented in [3] and [4] originally in the case of $3+1$-dimensional black holes and discuss how they can be, in principle, generalized. Let us note that both these arguments, contrary to the original Bekenstein idea, relied upon the existence of black hole temperature and Hawking evaporation, which became known only after Bekenstein proposal [2]

2.1 Thermodynamics and Hawking evaporation.

The first argument [3] (see also [5]) used the first law of thermodynamics for black holes [18]

$$\delta M = \frac{1}{4} T\delta A + \Omega\delta J + \Phi\delta Q \tag{2.1}$$

where T, Ω and Φ are the temperature, rotational frequency and electric potential of the black hole with mass M, surface area A, angular momentum J and charge Q, and an assumption that Hawking evaporation can be described as the result of a spontaneous transition between discrete levels of black hole and it was assumed that transitions occur between nearest level, let say n and $n-1$. Then for radiation mode characterized by a frequency ω, a charge e and azimutal quantum number m one has

$$\omega - e\Phi - m\Omega = \alpha T \tag{2.2}$$

where $\alpha = ln2$ if one assumes [3], [5] that during each transition the emitted quantum carries a minimum quantum of information - one bit. Because $\omega = M_n - M_{n-1} = \delta M$ one can see immdeiately that $\delta A = A_n - A_{n-1} = 4\alpha$ and $A_n = 4\alpha n$ which predicts the mass spectrum of the Schwarzchild black hole ($\Omega = \Phi = 0$)

$$M_n \sim M_p \sqrt{n} \tag{2.3}$$

Let us note that we got this spectrum making the assumption that $\omega = M_n - M_{n-1} \sim T$. However one can imagine the situation when transitions between all levels are of the same order of magnitude and in this case the radiation temperature, i.e. the characterisitc energy emitted, may be much larger than the distance between initial energy level M_n and the closest one M_{n-1}. For example of somebody gets the mass spectrum $M_n \sim M_p n$ and $T_n \sim M_p \sqrt{n}$ (as we shall see later this is true for $2+1$ dimensional black holes), i.e.

at large n one has $T >> \delta M = M_n - M_{n-1}$ and to avoid the paradox one must assume that there are unsuppressed transition matrix elements $< n|m >$ for all $|n-m| \leq \sqrt{n}$. Then the average energy which may be emitted will be by order of the maximal possible level splitting, i.e. $M_{n+\sqrt{n}} - M_n \sim M_p\sqrt{n} \sim T$ and one can try to reproduce the Hawking spectrum simply changing the pattern of the transition matrix elements.

2.2 String winding modes and left-right decoupling.

In [4] another approach to the quantization was suggested based on a picture of a test string in a black hole gravitational background

$$S = \frac{1}{2\pi}\int d^2\xi G_{\mu\nu}(x)\partial_a x^\mu \partial_a x^\nu \tag{2.4}$$

where the Schwarrzschild metric (after analytical continuation into the Euclidean region) takes the form

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu = \left(1-\frac{r_+}{r}\right)dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.5}$$

where r_+ is a horison radius proportional to the black hole mass $M = M_p^2 r_+$. This metric describes a regular manifold with the topology $R^2 \times S^2$ provided $r \geq r_+$ and imaginary time t is an angular variable with periodicity $4\pi r_+$, which means that we have the temperature $T = 1/4\pi r_+ = M_p^2/4\pi M$. This periodicity can be seen by making the substitution $r = r_+ + y^2/4r_+$ in the limit $r \to r_+$, then metric becomes $ds^2 = dy^2 + y^2 d(t/2r_+)^2 + r_+^2 d\Omega^2$ and to avoid the conical singularity at $y = 0$ one has to make $t/2r_+$ angular variable with a period 2π.

Let us examine the equations of motion which follows from (2.4)

$$\partial_z\partial_{\bar{z}}x^\mu + \Gamma^\mu_{\nu\rho}(x)\partial_z x^\nu \partial_{\bar{z}} x^\rho = 0 \tag{2.6}$$

where $z = \sigma + \tau$ and $\bar{z} = \sigma - \tau$. It is easy to see that there are exact left- or right-moving solutions of the form

$$x_L = x(z) = x(\tau + \sigma), \quad x_R = x(\bar{z}) = x(\tau - \sigma) \tag{2.7}$$

Now let us ask the question if these chiral solutions exist for general background, i.e. for general horison radius r_+ which, in our case, determines the metric. Expanding chiral (let say left) solution in oscillators

$$x_L(\tau + \sigma) = x_L + p_L(\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} a_n \exp\left[2in(\tau + \sigma)\right] \tag{2.8}$$

If one demands the absence of the right mode we are dealing with the winding mode where p_L is the winding number. One can ask the question how it is possible to consider only left modes and have no right mode at all. Generally speaking both sector - left and right - exist in a given background. However in case of Eucledian metric of black hole one must remember that it is only the part of the space-time with $r > r_+$ which has been analytically continued. One can imagine the situation (not in string theory, but in topological membrane where left- and right world sheets are independent) when left sector is above the horison and right one is beyond. In this case after analytical continuation we indeed have only left movers.

It is easy to see that it is impossible to have nonzero p_L in general case - because we are dealing with closed strings $x_L^\mu(\tau) = x_L^\mu(\tau + 2\pi)$ which excludes any nonzero p_L^μ. However this is not correct for $x^0 = t$ mode. Because of periodicity it is possible to have $x_L^0(\tau) = x_L^0(\tau + 2\pi) + 4\pi m r_+$ where m is some integer - winding number. This gives us $p_L^0 \sim m r_+$. However because p_L^0 is the momentum conjugate to the periodic variable it spectrum must be $p_L^0 \sim n/r_+$ where n is another integer. So in special case $r_+^2 \sim n/m$ we have a loophole. Assuming the "main" spectrum corresponds to winding number $m = 1$

(the only stable state - all others with $m > 1$ can decay into states with $m = 1$) we get the discrete spectrum $r_+^2 \sim n$, i.e. the same spectrum $M \sim M_p\sqrt{n}$

Thus we see that the same mass spectrum can be obtained from two absolutely different approaches. It will be interesting to understand if they will lead to equivalent predicitions in other cases. In the next section we shall consider $2+1$-dimensional black holes and will find that one again will have $r_+^2 \sim n$.

3 Quantum Spectrum of 2+1 Black Holes.

Let us consider $2+1$ Einstein gravity coupled to abelian topologically massive gauge field defined by action

$$S = \int d^3x \left\{ \frac{1}{\kappa}\sqrt{-g}\left[R + 2l^{-2}\right] \frac{1}{2}\sqrt{-g}F_{\mu\nu}F^{\mu\nu} - m\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda + \sqrt{-g}J^\mu A_\mu \right\} \tag{3.9}$$

where l^{-2} is the cosmological constant, $\kappa = 1/16\pi G$ is the Planck mass and J^μ is the covariantly conserved current: $D_\mu J^\mu = 0$. The coupled Einstein-Maxwell equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \kappa T_{\mu\nu} + g_{\mu\nu}l^{-2} \\ \partial_\nu(\sqrt{-g}F^{\nu\sigma}) + m\epsilon^{\sigma\mu\nu}F_{\mu\nu} &= \sqrt{-g}J^\sigma \end{aligned} \tag{3.10}$$

where the stress-energy tensor $T_{\mu\nu}$ does not depend on the gauge Chern-Simons term and equals to $T_{\mu\nu} = -F_{\mu\rho}F_\nu^\rho + \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}$

We shall consider solutions which depend only on radial coordinate r only, then the metric can be represented as

$$ds^2 = -N^2(r)dt^2 + dr^2 + R^2(r)d\theta^2 \tag{3.11}$$

and only non-zero $F_{\mu\nu}$ components are electric $F_{0r} = E(r)$ and magnetic $F_{r\theta} = H(r)$ fields. It is easy to see that from $D_\mu J^\mu = 0$ one gets $J^r = 0$. After simple calculations

one gets ($X' = dX/dr$):

$$\begin{aligned} R_{00} &= NN'' + NN'\frac{R'}{R} = \kappa\frac{N^2}{R^2}H^2 + 2l^{-2}N^2 \\ R_{rr} &= -\frac{N''}{N} - \frac{R''}{R} = -2l^{-2} \\ R_{\theta\theta} &= -RR'' - RR'\frac{N'}{N} = \kappa\frac{R^2}{N^2}E^2 - 2l^{-2}R^2 \end{aligned} \tag{3.12}$$

and

$$\frac{d}{dr}(\frac{R}{N}E) + mH = RNJ^0, \qquad \frac{d}{dr}(\frac{N}{R}H) + mE = RNJ^\theta \tag{3.13}$$

It is easy to see from (3.13), (3.13) that Eistein-Maxwell equations are symmetric under the transformation

$$N \leftrightarrow R, \;\; E \leftrightarrow H, \;\; J^\theta \leftrightarrow J^0, \;\; \kappa \rightarrow -\kappa \tag{3.14}$$

which is easy to understand because of the formal symmetry between θ and it in (3.11).

However, one can put $J^\theta = 0$ and get two solutions with $E \sim N/R, \; H = 0$ and $H \sim R/N, \; E = 0$ considering point-like charge in pure Maxwell theory, i.e. Chern-Simons mass term is zero, $m = 0$ or uniform charge distribution in the topologically massive gauge theory with non-zero Chern-Simons term $m \neq 0$. Then we see that the abovementioned duality can be realised not as duality betwen time and angular components of the current in the same theory, but as a duality between point-like charge distribution in pure Maxwell theory and uniform charge distribution in topologically massive gauge theory.

In [12] the case $m = 0, \; l^{-2} \neq 0$ has been considered and in [14] the opposite case case with zero cosmological constant $l^{-2} = 0$ and nonzero m. Let us consider the neutral black hole [12].

$$\frac{N''}{N} + \frac{N}{N'}\frac{R'}{R} = 2l^{-2}, \quad \frac{N''}{N} + \frac{R''}{R} = 2l^{-2}, \quad \frac{R''}{R} + \frac{R'}{R}\frac{N'}{N} = 2l^{-2} \tag{3.15}$$

from which one gets

$$\frac{R''}{R} = l^{-2}, \quad N = R' \tag{3.16}$$

The black hole metric can be rewritten in the form [12]

$$ds^2 = -N^2 dt^2 + \frac{dR^2}{N^2} + R^2 d\theta^2 \tag{3.17}$$

where

$$R = R_+ \cosh(r/l), \quad N = R' = \sqrt{(R^2 - R_+^2) l^2} \tag{3.18}$$

and R_+ is the radius of the horison. The mass of the black hole (in Planck units) is $M = (R_+/l)^2$ and one can find Hawking temperature using the analytical continuation $t \to it$. Then to have regular manifold with $R \leq R_+$ one has to make t periodic variable with period $2\pi l^2/R_+$ and the temperature is $T = R_+/2\pi l^2 = \sqrt{M}/2\pi l$. The manifold has topologu $R^2 \times S^1$, where S^1 corresponds to angle θ.

Using the same arguments about decoupling of left- and right moving modes (but in this case for angle coordinate θ, not t) one can conclude that quantization condition must be $R_+^2 \sim n$ and thus mass must be quantized as $M \sim n$, not $\sqrt{n}$. Moreover spectrum for R_+^2 (but not for mass M) must be independent on cosmological constant l^2.

It is amusing that this argument is supported by Carlip results [16] who used a Chern-Simons [19], [20] description for the 2 + 1 gravity. Using the fact the horison acts as a boundary he got an effective boundary dynamics describing the dynamical degrees of freedom on horison (for details see his paper) and found that the entropy of black hole $S = 2\pi R_+/4G$ is equal to the logarithm of number of boundary states. To be more precise, he found that the Virasoro operator on the boundary is

$$L_0 = N - (R_+/4G)^2 \tag{3.19}$$

where N is a number operator. Using the fact that the number of states for given N behaves as $n(N) \sim exp(\pi\sqrt{4N})$ he found that entropy

$$S = lnn(N) = 2\pi\sqrt{N} = 2\pi R_+/4G \tag{3.20}$$

which is the correct expression for the entropy.

However, this means at the same time that Virasoro constarint

$$L_0 = 0 \tag{3.21}$$

means the quantization condition for the black hole

$$\frac{R_+^2}{16G^2} = N \tag{3.22}$$

- the same as one can get from the abovementioned argument. Let us also note that there is no dependence on l^{-2} - which means that this quantum spectrum can not be obtained from any purely thermodynamical arguments (because both mass and tempearture depends explicitly on cosmological constant).

One can also consider the case of the black holes interacting with the TMGT. IN this case one can get additional contribution to the Virasoro operator due to the induced gauge degrees of freedom at the horison. For abelian Chern-Simons theory this contribution to the Virasoro generator will be proportional to A_θ^2 and using the solution with the uniform density of charge which has been considered in [14] one can get the spectrum in this case.

4 Conclusion

In this letter we tried to compare some handwaving arguments in favour of discrete spectrum of black holes and demonstrate that at least in one case - $2+1$ dimensional black

holes - one can obtain this spectrum. It will be extremely interesting to understand if the same spectrum can be reproduced in membrane approach advocated by Maggiore [7], [15] - and if there is any connection between spectral condiiton in his approach (existence of normalizable wave function for fluctuating membrane) and existence of chiral string modes.

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